Exercise 2.5.18

If the mass density is constant, using the result of Exercise 2.5.17, show that $\nabla \cdot \mathbf{u} = 0$.

Solution

Start with the continuity equation.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0$$

If ρ is constant, then the time derivative $\partial \rho / \partial t$ is zero.

$$\nabla \cdot \rho \mathbf{v} = 0$$

$$\nabla \cdot \rho \langle v_1, v_2, v_3 \rangle = 0$$

$$\nabla \cdot \langle \rho v_1, \rho v_2, \rho v_3 \rangle = 0$$

Evaluate the divergence.

$$\frac{\partial}{\partial x}(\rho v_1) + \frac{\partial}{\partial y}(\rho v_2) + \frac{\partial}{\partial z}(\rho v_3) = 0$$

Bring the constants in front.

$$\rho \frac{\partial v_1}{\partial x} + \rho \frac{\partial v_2}{\partial y} + \rho \frac{\partial v_3}{\partial z} = 0$$

Divide both sides by ρ.

$$\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = 0$$

Therefore,

$$\nabla \cdot \mathbf{v} = 0.$$